

Mass renormalization in nonrelativistic QED

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Abstract

In nonrelativistic QED the charge of an electron equals its bare value, whereas the self-energy and the mass have to be renormalized. In our contribution we study perturbative mass renormalization, including second order in the fine structure constant α , in the case of a single, spinless electron. As well known, if m denotes the bare mass and m_{eff} the mass computed from the theory, to order α one has

$$\frac{m_{\text{eff}}}{m} = 1 + \frac{8\alpha}{3\pi} \log(1 + \frac{1}{2}(\Lambda/m)) + O(\alpha^2)$$

which suggests that $m_{\text{eff}}/m = (\Lambda/m)^{8\alpha/3\pi}$ for small α . If correct, in order α^2 the leading term should be $\frac{1}{2}((8\alpha/3\pi) \log(\Lambda/m))^2$. To check this point we expand m_{eff}/m to order α^2 . The result is $\sqrt{\Lambda/m}$ as leading term, suggesting a more complicated dependence of m_{eff} on m .

1 Introduction

Nonperturbative renormalization in relativistic QED remains as a mathematical challenge. Thus it is of interest to study simplified candidates, an obvious one being nonrelativistic QED. In this theory, with comparable little effort, one can start from a self-adjoint Hamiltonian operator and thus has a well-defined mathematical framework. As an additional simplification, there is no charge renormalization because of the absence of positrons. Nevertheless, even in nonrelativistic QED, energy and mass renormalization remain poorly understood. Our, admittedly modest, contribution is to study mass renormalization including order α^2 .

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Let us first explain the basic Hamiltonian. We consider a single, spinless free electron coupled to the quantized radiation field. We will use relativistic units and employ immediately the total momentum representation. Then the Hilbert space of states is the symmetric Fock space, \mathcal{F} , over the one-particle space $L^2(\mathbb{R}^3 \times \{1, 2\})$, i.e.

$$\mathcal{F} = \oplus_{n=0}^{\infty} \otimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\}).$$

The inner product in \mathcal{F} is denoted by (\cdot, \cdot) and the Fock vacuum by Ω . On \mathcal{F} we introduce the Bose field

$$a(f) = \sum_{j=1,2} \int f(k, j)^* a(k, j) dk, \quad f \in L^2(\mathbb{R}^3 \times \{1, 2\}). \quad (1.1)$$

Operators $a(f)$ and $a(f)^* = a^*(f)$ are densely defined and satisfy the CCR

$$\begin{aligned} [a(f), a^*(g)] &= (f, g)_{L^2(\mathbb{R}^3 \times \{1, 2\})}, \\ [a(f), a(g)] &= 0, \\ [a^*(f), a^*(g)] &= 0. \end{aligned}$$

The kinetic energy of the photon is given by

$$H_f = \sum_{j=1,2} \int \omega(k) a^*(k, j) a(k, j) dk, \quad (1.2)$$

which is the second quantization of $\omega(k) = |k|$ considered as a multiplication operator on $L^2(\mathbb{R}^3)$. Similarly the momentum of the photon field is

$$P_f = \sum_{j=1,2} \int k a^*(k, j) a(k, j) dk. \quad (1.3)$$

The coupling of the electron to the Maxwell field is mediated through the transverse vector potential $A_{\hat{\varphi}}$ defined by

$$A_{\hat{\varphi}} = \frac{1}{\sqrt{2}}(a(f) + a^*(f)), \quad (1.4)$$

where

$$f(k, j) = \frac{1}{\sqrt{\omega}} \hat{\varphi}(k) e(k, j) \quad (1.5)$$

with $k/|k|, e(k, 1), e(k, 2)$ forming a right-handed dreibein in \mathbb{R}^3 . $\hat{\varphi}$ is the form factor which, as a minimal assumption, satisfies

$$\int_{\mathbb{R}^3} |\hat{\varphi}(k)|^2 (\omega(k)^{-2} + \omega(k)) dk < \infty. \quad (1.6)$$

Later on, we will make more specific choice of $\hat{\varphi}$.

With these definitions the Hamiltonian under study is

$$H_{\hat{\varphi}}(p) = \frac{1}{2} : (p - P_f - eA_{\hat{\varphi}})^2 : + H_f, \quad p \in \mathbb{R}^3, \quad (1.7)$$

where p is the total momentum, e the charge, to be definite $e \geq 0$, and $:X:$ denotes the Wick order of X . $H_{\hat{\varphi}}(p)$ with domain $D(H_f + \frac{1}{2}P_f^2) = D(H_f) \cap D(\frac{1}{2}P_f^2)$ is self-adjoint for e and p with $|e| < e_0$ and $|p| < p_0$ for some e_0 and p_0 , provided (1.6) holds. The energy-momentum relation is defined as the bottom of the spectrum of $H_{\hat{\varphi}}(p)$,

$$E_{\hat{\varphi}}(p) = \inf \sigma(H_{\hat{\varphi}}(p)). \quad (1.8)$$

In (1.7) the bare mass m of the electron is still missing. It appears in two places. Firstly the form factor depends on m . Let us assume a sharp ultraviolet cutoff Λ . Then

$$\begin{aligned} \hat{\varphi}(k) &= \hat{\varphi}_0(mck/\Lambda), \quad \Lambda > 0, \\ \hat{\varphi}_0(k) &= \begin{cases} (2\pi)^{-3/2} & \text{for } |k| \leq 1, \\ 0 & \text{for } |k| > 1, \end{cases} \end{aligned} \quad (1.9)$$

with $1/mc$ the Compton wave length. Secondly energy is to be measured in units of mc^2 and momentum in units of mc . We henceforth set $c = 1$ (and also $\hbar = 1$). Thus the true energy-momentum relation of the Pauli-Fierz Hamiltonian is

$$E_{m,\Lambda}(p) = mE_{\hat{\varphi}}(p/m), \quad \hat{\varphi} \text{ of (1.9)}. \quad (1.10)$$

Note that equivalently $E_{m,\Lambda}(p)$ is given through

$$E_{m,\Lambda}(p) = \inf \sigma\left(\frac{1}{2m} : (p - P_f - eA_{\hat{\varphi}_0(\cdot/\Lambda)})^2 : + H_f\right).$$

Removal of the ultraviolet cutoff Λ through mass renormalization means to find sequences

$$\Lambda \rightarrow \infty, \quad m \rightarrow 0 \quad (1.11)$$

such that $E_{m,\Lambda}(p) - E_{m,\Lambda}(0)$ has a nondegenerate limit. A convenient criterion for nondegeneracy is the curvature of $E_{m,\Lambda}(p)$ at $p = 0$, in other words the inverse effective mass. Let us assume for a moment an infrared cutoff

$$\hat{\varphi}(k) = 0 \quad \text{for } |k| < \kappa/m$$

with some $0 < \kappa$. Then it is known [1] that, for $|e| < e_*$, $|p| < p_*$ with suitable $e_* > 0$ and $p_* > 0$, $H_{\hat{\varphi}}(p)$ has a nondegenerate ground state $\psi_g(p)$ separated by a gap from the continuum, i.e.

$$H_{\hat{\varphi}}(p)\psi_g(p) = E_{\hat{\varphi}}(p)\psi_g(p), \quad \psi_g(p) \in \mathcal{F},$$

has a unique solution. Let us set

$$E_{m,\Lambda}(p) - E_{m,\Lambda}(0) = \frac{1}{2m_{\text{eff}}}p^2 + \mathcal{O}(|p|^3) \quad (1.12)$$

for small p . Then, using second order perturbation theory in (1.10), one obtains

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{(\psi_g(0), (P_{\text{f}} + eA_{\hat{\varphi}})_{\mu} (H_{\hat{\varphi}}(0) - E_{\hat{\varphi}}(0))^{-1} (P_{\text{f}} + eA_{\hat{\varphi}})_{\mu} \psi_g(0))}{(\psi_g(0), \psi_g(0))}. \quad (1.13)$$

We assume that this formula remains valid also when $\kappa = 0$.

On the basis of (1.13), mass renormalization can be discussed more precisely. From (1.13) it trivially follows that m/m_{eff} depends only on the ratio Λ/m . It is convenient to express this dependence in the form

$$\frac{m_{\text{eff}}}{m} = h(\Lambda/m). \quad (1.14)$$

Clearly $h \geq 1$ and $h(0) = 1$. Let us set

$$\lambda = \frac{\Lambda}{m} \quad (1.15)$$

One expects that h is increasing in λ , because with increasing Λ more photons are bound to the electron which makes m_{eff} larger.

Let us distinguish several cases. If h has a finite limit as $\lambda \rightarrow \infty$, then the mass renormalization is finite,

$$m_{\text{eff}} = h(\infty)m.$$

Such kind of behavior occurs in the Nelson model [2]. Secondly let us consider the case that $h(\lambda)$ increases linearly for large λ . We set

$$h(\lambda) = 1 + b_0\lambda$$

with $b_0 > 0$. Then

$$m_{\text{eff}} = m + b_0\Lambda. \quad (1.16)$$

Hence mass renormalization is additive. This behavior is found in the dipole approximation to the Pauli-Fierz Hamiltonian, e.g. [3], and in the classical Abraham model [4]. If $m_{\text{eff}} > 0$ is imposed, then as $\Lambda \rightarrow \infty$ necessarily $m \rightarrow -\infty$. As soon as $m < 0$, $mH_{\hat{\varphi}}(p/m)$ is not bounded from below. Therefore we regard the theory as not renormalizable. Thus the case of interest is when for large λ

$$h(\lambda) \simeq b_0 \lambda^\gamma, \quad b_0 > 0, \quad 0 < \gamma < 1, \quad (1.17)$$

which defines the scaling exponent γ and the amplitude b_0 . γ depends on e , as does b_0 . Inserting (1.17) in (1.14), one obtains for sufficiently large λ ,

$$\frac{m_{\text{eff}}}{m} \simeq b_0 \left(\frac{\Lambda}{m} \right)^\gamma. \quad (1.18)$$

Thus the choice

$$m = \Lambda^{-\gamma/(1-\gamma)} b_1^{1/(1-\gamma)} \quad (1.19)$$

yields

$$\lim_{\Lambda \rightarrow \infty} m_{\text{eff}}(\Lambda) = m^* = b_0 b_1. \quad (1.20)$$

Here b_0 is fixed by $h(\lambda)$ and b_1 is a parameter which can be adjusted to yield the experimentally determined mass of the electron.

Of course, the difficulty with our discussion is that, while the scaling function is well defined, at present we have no technique to find out its behavior for large λ . For that reason we turn to perturbative renormalization which, through the interchange of the limits $\Lambda \rightarrow \infty$ and $e \rightarrow 0$, tries to guess the proper value of γ . The details will be given in the following sections, but let us summarize briefly our findings. The fine structure constant is defined through

$$\alpha = \frac{e^2}{4\pi}. \quad (1.21)$$

To first order one finds

$$h(\lambda) = 1 + \frac{8\alpha}{3\pi} \log(1 + \frac{1}{2}\lambda) + \mathcal{O}(\alpha^2), \quad (1.22)$$

which suggests

$$h(\lambda) \simeq \lambda^{8\alpha/3\pi} \quad (1.23)$$

for sufficiently large λ and therefore

$$\gamma = \frac{8\alpha}{3\pi}, \quad \alpha \ll 1. \quad (1.24)$$

To have a control check, one assumes that to second order

$$\begin{aligned} h(\lambda) &\simeq \lambda^{(8/3\pi)\alpha + b\alpha^2} \\ &\simeq 1 + \frac{8\alpha}{3\pi} \log \lambda + \frac{1}{2} \left(\frac{8\alpha}{3\pi} \log \lambda \right)^2 + b\alpha^2 \log \lambda + \mathcal{O}(\alpha^3) \end{aligned} \quad (1.25)$$

for small α . Therefore by expanding m_{eff}/m to order α^2 , one should find a term $(\log \lambda)^2$ with an already determined prefactor and a term proportional to $\log \lambda$, together with lower order terms. As to explained, this guess is not confirmed. *Instead* we prove that

$$h(\lambda) = 1 + \frac{8\alpha}{3\pi} \log(1 + \frac{1}{2}\lambda) + c_0 \alpha^2 \sqrt{\lambda} + \mathcal{O}(\alpha^3), \quad c_0 > 0, \quad (1.26)$$

for $|\alpha|$ small enough *depending on* Λ , which could suggest $\gamma = \frac{1}{2}$ independent of e .

Note added in proof: Since the completion of this work F.H. and K. R. Ito [5] extended the investigation to include the spin of the electron. The number of terms in the perturbation series up to the same order as studied here is then multiplied by a factor of 4. As a net result one finds that the leading divergence is proportional to Λ^2 , rather than $\Lambda^{1/2}$. Because of the interaction with the quantized magnetic field the effective mass (at the order considered) is more strongly ultraviolet divergent when spin is included.

Some aspects of the effective mass and its renormalization have been studied before. Spohn [6] investigates the effective mass of the Nelson model [2] from a functional integral point of view. Lieb and Loss [7, 8] study mass renormalization and binding energies for various models of matter coupled to the radiation field including the Pauli-Fierz model. Hainzl [9] and Hainzl and Seiringer [10] compute the leading order of the effective mass of the Pauli-Fierz Hamiltonian with spin 1/2.

Our paper is organized in the following way. In Section 2 we review under which conditions $E_{\hat{\varphi}}(p) = E_{\hat{\varphi}}(p, e)$ is jointly analytic in p and e . In Section 3 we set up the perturbation theory for the effective mass and work out explicitly the terms including α^2 . Their asymptotics with $\Lambda \rightarrow \infty$ is studied in Section 4.

2 Ground state and its analytic properties

Throughout this paper we assume that

$$\hat{\varphi}(k) = \begin{cases} 0 & \text{for } |k| < \kappa/m, \\ (2\pi)^{-3/2} & \text{for } \kappa/m \leq |k| \leq \Lambda/m, \\ 0 & \text{for } |k| > \Lambda/m. \end{cases}$$

For notational convenience, we shall use notations $H(p)$, A and $E(p)$ instead of $H_{\hat{\varphi}}(p)$, $A_{\hat{\varphi}}$ and $E_{\hat{\varphi}}(p)$, respectively.

Let \mathcal{F}_{κ} (resp. $\mathcal{F}_{\kappa,0}$) be the symmetric Fock space over $L^2(\mathbb{R}_{\kappa/m}^3 \times \{1, 2\})$ (resp. $L^2(\mathbb{R}_{\kappa/m}^{3\perp} \times \{1, 2\})$), where $\mathbb{R}_{\kappa/m}^3 = \{k \in \mathbb{R}^3 \mid |k| \geq \kappa/m\}$. Then it follows that

$$\mathcal{F} \cong \mathcal{F}_{\kappa} \otimes \mathcal{F}_{\kappa,0}. \quad (2.1)$$

It is seen that \mathcal{F}_{κ} reduces $H(p)$ and, under the identification (2.1),

$$H(p) \cong (H(p)|_{\mathcal{F}_{\kappa}}) \otimes 1 + 1 \otimes (H|_{\mathcal{F}_{\kappa,0}}). \quad (2.2)$$

The bottom of the continuous spectrum of $H(p)|_{\mathcal{F}_{\kappa}}$ is denoted by $E_c(p)$. Note that $\inf \sigma(H(p)|_{\mathcal{F}_{\kappa}}) = E(p)$. The following lemma can be proven in the similar manner as in [11].

Lemma 2.1 [11] *There exists a constant $p_* > 0$ independent of e with $|e| < e_0$ such that for $p \in \mathbb{R}^3$ with $|p| < p_*$,*

$$E_c(p) - E(p) > 0.$$

In particular there exists a ground state $\psi_{g,\kappa}(p) \in \mathcal{F}_{\kappa}$ of $H(p)|_{\mathcal{F}_{\kappa}}$ for $p \in \mathbb{R}^3$ provided $|p| < p_$.*

By Lemma 2.1, we see that $H(p)$ has the ground state

$$\psi_g(p) = \psi_{g,\kappa}(p) \otimes \Omega_{\kappa,0}$$

for $p \in \mathbb{R}^3$ provided $|p| < p_*$, where $\Omega_{\kappa,0}$ denotes the vacuum of $\mathcal{F}_{\kappa,0}$. To have uniqueness, one proves that for any ground state $\psi_g(p)$, one has

$$(\psi_g(p), \Omega) > 0$$

provided $|p| < p_*$ and $|e| < e_*$ with some e_* .

Lemma 2.2 [1] *There exists a constant $e_* > 0$ such that for $(p, e) \in \mathbb{R}^3 \times \mathbb{R}$ with $|p| < p_*$ and $|e| < e_*$, the ground state of $H(p)$ is unique up to multiple constants.*

Remark 2.3 *In the case $\kappa = 0$ and for sufficiently small e , Chen [12] proves the absence of a ground state of $H(p)$ in \mathcal{F} for $p \neq 0$ and the existence of a ground state of $H(0)$.*

We also need the analytic properties of $\psi_g(p) = \psi_g(p, e)$ and $E(p) = E(p, e)$ with respect to $(p, e) \in \mathbb{R}^3 \times \mathbb{R}$ in a neighborhood of $(0, 0) \in \mathbb{R}^3 \times \mathbb{R}$.

Lemma 2.4 *There exists an open neighborhood \mathcal{O} of $(0, 0) \in \mathbb{R}^3 \times \mathbb{R}$ such that $\psi_g(p, e)$ is strongly analytic and $E(p, e)$ analytic on \mathcal{O} .*

Proof: Let $\psi_{g,\kappa}(p) \in \mathcal{F}_\kappa$ be the ground state of $H(p) \upharpoonright_{\mathcal{F}_\kappa}$. Since $\psi_g(p) = \psi_{g,\kappa}(p) \otimes \Omega_{\kappa,0}$, it is enough to show that $\psi_{g,\kappa}(p)$ is strongly analytic on \mathcal{O} . We split $H(p)$ as

$$H(p) = H_0(p) + H_I(p), \quad (2.3)$$

where

$$\begin{aligned} H_0(p) &= \frac{1}{2}(p - P_f)^2 + H_f, \\ H_I(p) &= -e(p - P_f) \cdot A + e^2 \frac{1}{2} :A^2: . \end{aligned}$$

Then we obtain that

$$\|H_I(p)\Psi\|_{\mathcal{F}_\kappa} \leq c_4 \|H_0(p)\Psi\|_{\mathcal{F}_\kappa} + c_5 \|\Psi\|_{\mathcal{F}_\kappa} \quad (2.4)$$

for $\Psi \in D(H_0(p) \upharpoonright_{\mathcal{F}_\kappa}) = D(H_f) \cap D(P_f^2) \cap \mathcal{F}_\kappa$. Then $H(p) \upharpoonright_{\mathcal{F}_\kappa}$ is an analytic family of type (A) for e near $e = 0$ (see [13, p.16]). Thus by [13, Theorem XII.9], $H(p) \upharpoonright_{\mathcal{F}_\kappa}$ is an analytic family in the sense of Kato, which implies that by [13, Theorem XII.8], together with Lemmas 2.1 and 2.2, $\psi_{g,\kappa}(p, e)$ is strongly analytic and $E(p, e)$ analytic for e near $e = 0$. Alternatively we split $H(p)$ as

$$H(p) = H'_0 + p \cdot H'_I + \frac{1}{2}p^2,$$

where

$$H'_0 = \frac{1}{2}:(P_f + eA_{\hat{\varphi}})^2: + H_f, \quad H'_I = -(P_f + eA_{\hat{\varphi}}).$$

Then we have

$$\|H'_I \Psi\|_{\mathcal{F}_\kappa} \leq c_6 \|H'_0 \Psi\|_{\mathcal{F}_\kappa} + c_7 \|\Psi\|_{\mathcal{F}_\kappa} \quad (2.5)$$

with some constants c_6 and c_7 for $\Psi \in D(H_f) \cap D(P_f^2) \cap \mathcal{F}_\kappa$. Thus $H(p)|_{\mathcal{F}_\kappa}$ is an analytic family of type (A) for $p \in \mathbb{R}^3$ near $p = 0$. We can see that $\psi_{g,\kappa}(p, e)$ is strongly analytic and $E(p, e)$ analytic for p near $p = 0$ in the similar manner as for e . \square

3 Effective mass

3.1 Formulae

In what follows we assume that $(p, e) \in \mathcal{O}$. By the definition of $E(p, e)$, we have

$$\frac{m}{m_{\text{eff}}} = \frac{1}{3} \Delta_p E(p, e) \Big|_{p=0}. \quad (3.1)$$

Actually we can see in [1] that $H(p)$ is unitarily equivalent to $H(|p|n_z)$, where $n_z = (0, 0, 1)$. Thus $E(p, e) = \tilde{E}(|p|, e) = \inf \sigma(H(|p|n_z))$ and

$$\frac{m}{m_{\text{eff}}} = \partial_{|p|}^2 \tilde{E}(|p|, e) \Big|_{|p|=0}.$$

Moreover we see that $\tilde{E}(-|p|, e) = \tilde{E}(|p|, e)$. Then

$$\partial_{p_\mu} E(p, e) \Big|_{p_\mu=0} = 0, \quad \mu = 1, 2, 3. \quad (3.2)$$

Since $E(p, e)$ also has the symmetry, $E(p, -e) = E(p, e)$, $E(p, e)$ is a function of e^2 . In particular it follows that

$$\partial_e^{2m+1} E(p, e) \Big|_{e=0} = 0, \quad m \geq 0. \quad (3.3)$$

Lemma 3.1 *We have*

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{(\psi_g(0), (P_f + eA)_\mu (H(0) - E(0))^{-1} (P_f + eA)_\mu \psi_g(0))}{(\psi_g(0), \psi_g(0))}.$$

Proof: Since

$$(H(p)\Psi, \psi_g(p)) = E(p)(\Psi, \psi_g(p)),$$

for $\Psi \in D(H(p))$, taking a derivative with respect to p_μ on the both sides above, we have

$$(H'_\mu(p)\Psi, \psi_g(p)) + (H(p)\Psi, \psi'_{g_\mu}(p)) = E'_\mu(p)(\Psi, \psi_g(p)) + E(p)(\Psi, \psi'_{g_\mu}(p)) \quad (3.4)$$

and

$$\begin{aligned} & (H''_\mu(p)\Psi, \psi_g(p)) + 2(H'_\mu(p)\Psi, \psi'_{g\mu}(p)) + (H(p)\Psi, \psi''_{g\mu}(p)) \\ &= E''_\mu(p)(\Psi, \psi_g(p)) + 2E'_\mu(p)(\Psi, \psi'_{g\mu}(p)) + E(p)(\Psi, \psi''_{g\mu}(p)). \end{aligned} \quad (3.5)$$

Here $E'_\mu(p)$ (resp. $\psi'_{g\mu}(p)$) denotes the derivative (resp. strong derivative) in p_μ , and $H'_\mu(p) = (p - P_f - eA_{\hat{\varphi}})_\mu$, $H''_\mu(p) = 1$. By (3.2) it follows that $E'_\mu(0) = 0$, and by (3.4) with $p = 0$,

$$(P_f + eA)_\mu \psi_g(0) \in D((H(0) - E(0))^{-1}),$$

and

$$\psi'_{g\mu}(0) = (H(0) - E(0))^{-1}(P_f + eA)_\mu \psi_g(0).$$

Therefore, using (3.1) and (3.5), we have

$$\begin{aligned} \frac{m}{m_{\text{eff}}} &= \frac{1}{3} \sum_{\mu=1,2,3} \frac{(\psi_g(0), E''(0)_\mu \psi_g(0))}{(\psi_g(0), \psi_g(0))} \\ &= \frac{1}{3} \sum_{\mu=1,2,3} \left\{ 1 + \frac{(\psi_g(0), 2H'_\mu(0)\psi'_{g\mu}(0))}{(\psi_g(0), \psi_g(0))} \right\} \\ &= 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{((P_f + eA)_\mu \psi_g(0), (H(0) - E(0))^{-1}(P_f + eA)_\mu \psi_g(0))}{(\psi_g(0), \psi_g(0))}. \end{aligned} \quad (3.6)$$

Thus the lemma follows. \square

From this lemma we obtain the following corollary.

Corollary 3.2 *Let $|e| < e_*$. Then $m_{\text{eff}} \geq m$.*

3.2 Perturbative expansions

Let

$$\psi_g(0) = \sum_{n=0}^{\infty} \frac{e^n}{n!} \varphi_n, \quad E(0) = \sum_{n=0}^{\infty} \frac{e^{2n}}{(2n)!} E_{2n}.$$

We want to get the explicit form of φ_n . Let

$$\begin{aligned} \mathcal{F}_{\text{fin}} &= \{ \{ \Psi^{(n)} \}_{n=0}^{\infty} \in \mathcal{F} \mid \Psi^{(m)} = 0 \text{ for } m \geq \ell \text{ with some } \ell \}, \\ \mathcal{F}_0 &= \{ \Psi \in \mathcal{F}_{\text{fin}} \mid \text{(i) } \Psi^{(0)} = 0, \text{ (ii) } \text{supp}_{k \in \mathbb{R}^{3n}} \Psi^{(n)}(k, j) \not\supset \{0\}, n \geq 1, j \in \{1, 2\}^n \}. \end{aligned}$$

Lemma 3.3 *We see that $\mathcal{F}_0 \subset D(H_0^{-1})$.*

Proof: Let $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_0$. Since

$$\begin{aligned} & (H_0^{-1}\Psi)^{(n)}(k_1, \dots, k_n, j_1, \dots, j_n) \\ &= \left[\frac{1}{2}(k_1 + \dots + k_n)^2 + \sum_{i=1}^n \omega(k_i) \right]^{-1} \Psi^{(n)}(k_1, \dots, k_n, j_1, \dots, j_n), \end{aligned}$$

and $\text{supp}_{(k_1, \dots, k_n) \in \mathbb{R}^{3n}} \Psi^{(n)}(k_1, \dots, k_n, j_1, \dots, j_n) \not\supset \{(0, \dots, 0)\}$, we obtain that

$$\|H_0^{-1}\Psi\|_{\mathcal{F}}^2 = \sum_{n=1}^{\infty} \|(H_0^{-1}\Psi)^{(n)}\|_{\mathcal{F}^{(n)}}^2 < \infty$$

and the lemma follows. \square

We define A^- and A^+ by

$$A^- = \frac{1}{\sqrt{2}}a(f), \quad A^+ = \frac{1}{\sqrt{2}}a^*(f).$$

Then $A = A^+ + A^-$. Moreover A_μ^- and A_μ^+ are defined by A^- and A^+ with $e(k, j)$ replaced by $e^\mu(k, j)$. We split $H(0)$ as

$$H(0) = H_0 + eH_1 + \frac{e^2}{2}H_2,$$

where

$$\begin{aligned} H_0 &= \frac{1}{2}P_{\text{f}}^2 + H_{\text{f}}, \\ H_1 &= \frac{1}{2}(P_{\text{f}} \cdot A + A \cdot P_{\text{f}}) = P_{\text{f}} \cdot A = A \cdot P_{\text{f}}, \\ H_2 &=: A^2 := A^+ \cdot A^+ + A^- \cdot A^- + 2A^+ \cdot A^-. \end{aligned}$$

Lemma 3.4 *We have $E_0 = E_2 = 0$, and there exists a ground state $\psi_{\text{g}}(0) = \sum_{n=0}^{\infty} \frac{e^n}{n!} \varphi_n$ such that*

$$\varphi_0 = \Omega, \quad \varphi_1 = 0, \quad \varphi_2 = -H_0^{-1}H_2\Omega, \quad \varphi_3 = 3H_0^{-1}H_1H_0^{-1}H_2\Omega. \quad (3.7)$$

In particular $\varphi_2 \in \mathcal{F}^{(2)}$ and $\varphi_3 \in \mathcal{F}^{(1)} \oplus \mathcal{F}^{(3)}$.

Proof: It is obvious that $E_0 = 0$. Let $\psi_g(0) = \sum_{n=0}^{\infty} \frac{e^n}{n!} \varphi_n$ be an arbitrary strongly analytic ground state of $H(0)$ with $(\varphi_0, \Omega) \neq 0$. Let $\rho(e) = \sum_{n=0}^{\infty} \frac{e^n}{n!} \rho_n$ be an analytic function on e . Then $\rho\psi_g(0)$ is also a strongly analytic ground state of $H(0)$ and

$$\begin{aligned} \rho\psi_g(0) &= \underbrace{\rho_1\varphi_0}_{=\varphi_0^\rho} + e \underbrace{(\rho_0\varphi_1 + \rho_1\varphi_0)}_{=\varphi_1^\rho} + e^2 \underbrace{\frac{1}{2!}(\rho_0\varphi_2 + 2\rho_1\varphi_1 + \rho_2\varphi_0)}_{=\varphi_2^\rho} \\ &\quad + e^3 \underbrace{\frac{1}{3!}(\rho_0\varphi_3 + 3\rho_1\varphi_2 + 3\rho_2\varphi_1 + \rho_3\varphi_0)}_{=\varphi_3^\rho} + \mathcal{O}(e^4). \end{aligned}$$

Set

$$\begin{aligned} \rho_0 &= 1/(\varphi_0, \Omega), \quad \rho_1 = -\rho_0(\varphi_1, \Omega)/(\varphi_0, \Omega), \\ \rho_2 &= -(\rho_0(\varphi_2, \Omega) + 2\rho_1(\varphi_1, \Omega))/(\varphi_0, \Omega), \\ \rho_3 &= -(\rho_0(\varphi_3, \Omega) + 3\rho_1(\varphi_2, \Omega) + 3\rho_2(\varphi_1, \Omega))/(\varphi_0, \Omega). \end{aligned}$$

Then $\psi_g^\rho = \sum_{n=0}^{\infty} \frac{e^n}{n!} \varphi_n^\rho$ satisfies that

$$(\varphi_n^\rho, \Omega) = \delta_{0,n}, \quad n = 0, 1, 2, 3. \quad (3.8)$$

We reset ψ_g^ρ (resp. φ_n^ρ) with (3.8) as $\psi_g(0)$ (resp. φ_n). Let us write $H(0)$, $E(0)$ and $\psi_g(0)$ as H , E and ψ_g , respectively. Take derivative in e on the both sides of $(H\Psi, \psi_g) = E(\Psi, \psi_g)$, $\Psi \in D(H)$. Then we have

$$(H'\Psi, \psi_g) + (H\Psi, \psi_g') = E'(\Psi, \psi_g) + E(\Psi, \psi_g'), \quad (3.9)$$

$$(H''\Psi, \psi_g) + 2(H'\Psi, \psi_g') + (H\Psi, \psi_g'') = E''(\Psi, \psi_g) + 2E'(\Psi, \psi_g') + E(\Psi, \psi_g''), \quad (3.10)$$

$$\begin{aligned} &3(H''\Psi, \psi_g') + 3(H'\Psi, \psi_g'') + (H\Psi, \psi_g''') \\ &= E'''(\Psi, \psi_g) + 3E''(\Psi, \psi_g') + 3E'(\Psi, \psi_g'') + E(\Psi, \psi_g'''), \quad (3.11) \end{aligned}$$

where E' (resp. ψ_g') denotes the derivative (resp. strong derivative) in e , and $H' = P_f(P_f + eA)$ and $H'' = P_f \cdot A$. Put $\Psi = \Omega$ and $e = 0$ in (3.10). Then

$$0 = E_2(\Omega, \Omega), \quad (3.12)$$

which shows that $E_2 = 0$. From (3.9) with $e = 0$ it follows that

$$H_1\Omega + H_0\varphi_1 = 0,$$

from which it holds that $H_0\varphi_1 = 0$. Hence $\varphi_1 = b\Omega$ with some constant b . By (3.8) we have, however, $b = 0$. Then $\varphi_1 = 0$ follows. By (3.10) with $e = 0$, we have

$$H_2\Omega + H_0\varphi_2 = 0.$$

Since $H_2\Omega \in \mathcal{F}_0$, we see that by Lemma 3.3, $H_2\Omega \in D(H_0^{-1})$. Thus we have $\varphi_2 = -H_0^{-1}H_2\Omega + c\Omega$ with some constant c . Since $(-H_0^{-1}H_2\Omega, \Omega) = 0$, it follows that $c = 0$ from (3.8). From (3.11) it follows that in $e = 0$,

$$3H_1\varphi_2 + H_0\varphi_3 = 0.$$

Since $H_1\varphi_2 = -H_1H_0^{-1}H_2\Omega \in \mathcal{F}_0$, Lemma 3.3 ensures that $H_1\varphi_2 \in D(H_0^{-1})$. Hence $\varphi_3 = -3H_0^{-1}H_1\varphi_2 + d\Omega = 3H_0^{-1}H_1H_0^{-1}H_2\Omega + d\Omega$ with some constant d . Since $(3H_0^{-1}H_1H_0^{-1}H_2\Omega, \Omega) = 0$, it follows that $d = 0$ from (3.8). Then the lemma is proven. \square

In the similar manner as Lemma 3.4, we can prove the following proposition.

Proposition 3.5 *There exists a ground state $\psi_g(0) = \sum_{n=0}^{\infty} \frac{e^n}{n!} \varphi_n$ such that*

$$\begin{aligned} \varphi_{2m} &= H_0^{-1} \left\{ - \sum_{j=1,2} \binom{2m}{j} H_1 \varphi_{2m-j} + \sum_{j=2}^m \binom{2m}{2j} E_{2j} \varphi_{2m-2j} \right\}, \quad m \geq 2, \\ \varphi_{2m+1} &= H_0^{-1} \left\{ - \sum_{j=1,2} \binom{2m+1}{j} H_j \varphi_{2m+1-j} + \sum_{j=2}^{m-1} \binom{2m+1}{2j} E_{2j} \varphi_{2m-2j+1} \right\}, \quad m \geq 2, \end{aligned}$$

with $\varphi_{2m} \in \mathcal{F}^{(2)} \oplus \mathcal{F}^{(4)} \oplus \dots \oplus \mathcal{F}^{(2m)}$ and $\varphi_{2m+1} \in \mathcal{F}^{(1)} \oplus \mathcal{F}^{(3)} \oplus \dots \oplus \mathcal{F}^{(2m+1)}$, and E_{2m} is given by

$$E_{2m} = \binom{2m}{2} (\Omega, H_2 \varphi_{2m-2}), \quad m \geq 2.$$

3.3 Effective mass up to order e^4

In this subsection we expand m/m_{eff} up to order e^4 .

Lemma 3.6 *We have*

$$\begin{aligned} \frac{m}{m_{\text{eff}}} &= 1 - e^2 \frac{2}{3} \sum_{\mu=1,2,3} (\Omega, A_\mu H_0^{-1} A_\mu \Omega) \\ &\quad - e^4 \frac{2}{3} \sum_{\mu=1,2,3} \left\{ 2 (\Psi_3^\mu, H_0^{-1} \Psi_1^\mu) + (\Psi_2^\mu, H_0^{-1} \Psi_2^\mu) - 2 (\Psi_2^\mu, H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu) \right. \\ &\quad \left. - \frac{1}{2} (\Psi_1^\mu, H_0^{-1} H_2 H_0^{-1} \Psi_1^\mu) + (\Psi_1^\mu, H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu) \right\} + \mathcal{O}(e^6), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned}\Psi_1^\mu &= A_\mu \Omega, \\ \Psi_2^\mu &= -\frac{1}{2} P_{f\mu} H_0^{-1} (A^+ \cdot A^+) \Omega, \\ \Psi_3^\mu &= \frac{1}{2} \left\{ -A_\mu H_0^{-1} (A^+ \cdot A^+) \Omega + \frac{1}{2} P_{f\mu} H_0^{-1} (P_f \cdot A + A \cdot P_f) H_0^{-1} (A^+ \cdot A^+) \Omega \right\}.\end{aligned}$$

Proof: Since by (3.6),

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{((P_f + eA)_\mu \psi_g(0), \psi'_{g\mu}(0))}{(\psi_g(0), \psi_g(0))}, \quad (3.14)$$

where $\psi'_{g\mu}(0) = s - \partial_{p_\mu} \psi_g(p) \big|_{p=0}$, we expand $\psi'_{g\mu}(0)$ and $\psi_g(0)$ in e . Assume that $\psi_g(0) = \sum_{n=0}^{\infty} \frac{e^n}{n!} \varphi_n$ satisfies (3.7), i.e., $\varphi_0 = \Omega$, $\varphi_1 = 0$, $\varphi_2 = -H_0^{-1} H_2 \Omega$ and $\varphi_3 = 3H_0^{-1} H_1 H_0^{-1} H_2 \Omega$. We have

$$\begin{aligned}(P_f + eA)_\mu \psi_g(0) &= eA_\mu \Omega + e^2 \left(\frac{1}{2} P_{f\mu} \varphi_2 \right) + e^3 \left(\frac{1}{2} A_\mu \varphi_2 + \frac{1}{6} P_{f\mu} \varphi_3 \right) + \mathcal{O}(e^4) \\ &= e\Psi_1^\mu + e^2 \Psi_2^\mu + e^3 \Psi_3^\mu + \mathcal{O}(e^4).\end{aligned} \quad (3.15)$$

Note that by Proposition 3.5,

$$\varphi_0 \in \mathcal{F}^{(0)}, \varphi_2 \in \mathcal{F}^{(2)}, \varphi_3 \in \mathcal{F}^{(3)} \oplus \mathcal{F}^{(1)}, \varphi_4 \in \mathcal{F}^{(4)} \oplus \mathcal{F}^{(2)}.$$

In particular

$$\frac{1}{(\psi_g, \psi_g)} = 1 - e^4 \left(\frac{1}{2} \varphi_2, \frac{1}{2} \varphi_2 \right) - e^4 \left(\Omega, \frac{1}{24} \varphi_4 \right) + \mathcal{O}(e^6) = 1 - e^4 \frac{1}{4} (\varphi_2, \varphi_2) + \mathcal{O}(e^6). \quad (3.16)$$

Let

$$\psi'_{g\mu}(0) = \sum_{n=0}^{\infty} \frac{e^n}{n!} \Phi_n^\mu. \quad (3.17)$$

Since

$$((H(0) - E(0))\Psi, \psi'_{g\mu}(0)) = ((P_f + eA)_\mu \Psi, \psi_g(0)), \quad \Psi \in D(H(0)), \quad (3.18)$$

putting $e = 0$ on the both sides of (3.18), we have

$$H_0 \Phi_0^\mu = 0.$$

Then

$$\Phi_0^\mu = b_0 \Omega \quad (3.19)$$

with some constant b_0 . From taking derivative of the both sides of (3.18) at $e = 0$, we see that by (3.7)

$$\begin{aligned} H_0 \Phi_1^\mu &= A_\mu \Omega, \\ H_2 \Phi_0^\mu + 2H_1 \Phi_1^\mu + H_0 \Phi_2^\mu &= P_{f\mu} \varphi_2, \\ 3H_2 \Phi_1^\mu + 3H_1 \Phi_2^\mu + H_0 \Phi_3^\mu &= 3A_\mu \varphi_2 + P_{f\mu} \varphi_3. \end{aligned}$$

From them it follows that

$$\Phi_1^\mu = H_0^{-1} \Psi_1^\mu + b_1 \Omega, \quad (3.20)$$

$$\begin{aligned} \Phi_2^\mu &= H_0^{-1} (2\Psi_2^\mu - 2H_1 \Phi_1^\mu - H_2 \Phi_0^\mu) + b_2 \Omega, \\ &= 2H_0^{-1} (\Psi_2^\mu - H_1 H_0^{-1} \Psi_1^\mu) + (-b_0 H_0^{-1} H_2 \Omega + b_2 \Omega) \end{aligned} \quad (3.21)$$

$$\begin{aligned} \Phi_3^\mu &= H_0^{-1} (6\Psi_3^\mu - 3H_1 \Phi_2^\mu - 3H_2 \Phi_1^\mu) + b_3 \Omega, \\ &= 6H_0^{-1} (\Psi_3^\mu - H_1 H_0^{-1} \Psi_2^\mu + (H_1 H_0^{-1} H_1 H_0^{-1} - \frac{1}{2} H_2 H_0^{-1}) \Psi_1^\mu) \\ &\quad + (3b_0 H_0^{-1} H_1 H_0^{-1} H_2 \Omega - 3b_1 H_0^{-1} H_2 \Omega + b_3 \Omega), \end{aligned} \quad (3.22)$$

where b_1, b_2, b_3 are some constants. Here we used that $H_1 \Omega = 0$. By (3.14), (3.15), (3.16) and (3.17) we have

$$\begin{aligned} \frac{m}{m_{\text{eff}}} &= 1 - \frac{2}{3} \sum_{\mu=1}^3 \left\{ e^2 ((\Psi_1^\mu, \Phi_1^\mu) + (\Psi_2^\mu, \Phi_0^\mu)) \right. \\ &\quad \left. + e^4 \left(\frac{1}{6} (\Psi_1^\mu, \Phi_3^\mu) + \frac{1}{2} (\Psi_2^\mu, \Phi_2^\mu) + (\Psi_3^\mu, \Phi_1^\mu) \right) \right\} + \mathcal{O}(e^6). \end{aligned} \quad (3.23)$$

Substitute (3.20)-(3.22) into (3.23). No contribution of constants b_0, \dots, b_3 exists, i.e., we can directly see that

$$e^2 \{b_1(\Psi_1^\mu, \Omega) + b_0(\Psi_2^\mu, \Omega)\} = 0.$$

and

$$\begin{aligned} e^4 \left\{ \frac{1}{6} b_3(\Psi_1^\mu, \Omega) + \frac{1}{6} b_1(\Psi_1^\mu, -3H_0^{-1} H_2 \Omega) + \frac{1}{6} b_0(\Psi_1^\mu, 3H_0^{-1} H_1 H_0^{-1} H_2 \Omega) \right. \\ \left. + \frac{1}{2} b_2(\Psi_2^\mu, \Omega) + \frac{1}{2} b_0(\Psi_2^\mu, -H_0^{-1} H_2 \Omega) + b_1(\Psi_3^\mu, \Omega) \right\} = 0. \end{aligned}$$

Then the lemma follows. \square

Remark 3.7 By Lemma 3.1 we have seen that

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{2}{3} \sum_{\mu=1,2,3} \frac{((P_{\text{f}} + eA)_{\mu} \psi_{\text{g}}(0), (H(0) - E(0))^{-1} (P_{\text{f}} + eA)_{\mu} \psi_{\text{g}}(0))}{(\psi_{\text{g}}(0), \psi_{\text{g}}(0))}. \quad (3.24)$$

We “informally” expand $(H(0) - E(0))^{-1}$ as

$$\begin{aligned} (H(0) - E(0))^{-1} &= \sum_{n=0}^{\infty} \left(-H_0^{-1} \sum_{l=1}^{\infty} \frac{e^l}{l!} H_l \right)^n H_0^{-1} \\ &= \sum_{n=0}^{\infty} (-1)^n \sum_{k=n}^{\infty} e^k \sum_{\substack{l_1, \dots, l_n=1 \\ l_1 + \dots + l_n = k}}^k \frac{1}{l_1! \dots l_n!} H_0^{-1} H_{l_1} H_0^{-1} H_{l_2} \dots H_0^{-1} H_{l_n} H_0^{-1}. \end{aligned} \quad (3.25)$$

Here we set $H_j = \begin{cases} H_j, & j = 1, 2, \\ -E_j, & j \geq 3. \end{cases}$ Then we have

$$\begin{aligned} (H(0) - E(0))^{-1} &= H_0^{-1} - e H_0^{-1} H_1 H_0^{-1} \\ &\quad + e^2 \left(-\frac{1}{2} H_0^{-1} H_2 H_0^{-1} + H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \right) + \mathcal{O}(e^3). \end{aligned} \quad (3.26)$$

Substitute (3.26) into (3.24). Then the result coincides with (3.13).

3.4 Explicit expressions

For each $k \in \mathbb{R}^3$ let us define the projection $Q(k)$ on \mathbb{R}^3 by

$$Q(k) = \sum_{j=1,2} |e_j(k)\rangle \langle e_j(k)|.$$

We also set

$$m = 1,$$

since it can easily be reintroduced at the end of the computation. We set

$$\hat{\varphi}_j = \hat{\varphi}(k_j), \quad \omega_j = \omega(k_j), \quad Q(k_j) = Q_j, \quad j = 1, 2.$$

Let

$$\begin{aligned} \frac{1}{E_j} &= \frac{1}{|k_j|^2/2 + \omega_j}, \quad j = 1, 2, \\ \frac{1}{E_{12}} &= \frac{1}{|k_1 + k_2|^2/2 + \omega_1 + \omega_2}, \quad k_1, k_2 \in \mathbb{R}^3, \\ \frac{1}{F_j} &= \frac{1}{r_j^2/2 + r_j}, \quad j = 1, 2, \\ \frac{1}{F_{12}} &= \frac{1}{(r_1^2 + 2r_1 r_2 X + r_2^2)/2 + r_1 + r_2}, \quad r_1, r_2 \geq 0, \quad -1 \leq X \leq 1. \end{aligned}$$

Lemma 3.8 *We have*

$$\frac{m}{m_{\text{eff}}} = 1 - \alpha a_1(\Lambda, \kappa) - \alpha^2 a_2(\Lambda, \kappa) + \mathcal{O}(\alpha^3),$$

where

$$a_1(\Lambda, \kappa) = \frac{8}{3\pi} \log \left(\frac{\Lambda + 2}{\kappa + 2} \right) \quad (3.27)$$

and

$$\begin{aligned} a_2(\Lambda, \kappa) = & (4\pi)^2 \frac{2}{3} \iint d^3 k_1 d^3 k_2 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \times \\ & \times \left\{ - \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \frac{1}{E_{12}} (1 + s^2) + \left(\frac{1}{E_{12}} \right)^3 \frac{|k_1 + k_2|^2}{2} (1 + s^2) \right. \\ & + \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \left(\frac{1}{E_{12}} \right)^2 (k_1 \cdot k_2) (-1 + s^2) - \frac{1}{E_1} \frac{1}{E_2} (1 + s^2) \\ & \left. + \left(\frac{|k_1|^2}{E_1^2} + \frac{|k_2|^2}{E_2^2} \right) \frac{1}{E_{12}} (1 - s^2) + \frac{1}{E_1} \frac{1}{E_2} \frac{1}{E_{12}} (k_1 \cdot k_2) (-1 + s^2) \right\}, \end{aligned} \quad (3.28)$$

where $s = (\hat{k}_1, \hat{k}_2)$. Changing variables to polar coordinates we also have

$$\begin{aligned} a_2(\Lambda, \kappa) = & \frac{(4\pi)^2}{(2\pi)^6} \frac{2}{3} \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr_1 \int_{\kappa}^{\Lambda} dr_2 \pi r_1 r_2 \times \\ & \times \left\{ - \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \frac{1}{F_{12}} (1 + X^2) + \left(\frac{1}{F_{12}} \right)^3 \frac{r_1^2 + 2r_1 r_2 X + r_2^2}{2} (1 + X^2) \right. \\ & + \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \left(\frac{1}{F_{12}} \right)^2 r_1 r_2 X (-1 + X^2) - \frac{1}{F_1} \frac{1}{F_2} (1 + X^2) \\ & \left. + \left(\frac{r_1^2}{F_1^2} + \frac{r_2^2}{F_2^2} \right) \frac{1}{F_{12}} (1 - X^2) + \frac{1}{F_1} \frac{1}{F_2} \frac{1}{F_{12}} r_1 r_2 X (-1 + X^2) \right\}. \end{aligned} \quad (3.29)$$

Proof: Note that

$$\begin{aligned} a_1(\Lambda, \kappa) &= \frac{2}{3} (\sqrt{4\pi})^2 (A_{\mu}^+ \Omega, H_0^{-1} A_{\mu}^+ \Omega) \\ &= \frac{2}{3} (\sqrt{4\pi})^2 2 \int \frac{\hat{\varphi}(k)^2}{2\omega(k)} \frac{1}{|k|^2/2 + |k|} d^3 k \\ &= \frac{2}{3} (\sqrt{4\pi})^2 \frac{1}{(2\pi)^3} 4\pi \int_{\kappa}^{\Lambda} \frac{1}{r/2 + 1} dr \\ &= \frac{8}{3\pi} \log \left(\frac{\Lambda + 2}{\kappa + 2} \right). \end{aligned}$$

Thus (3.27) follows. To check $a_2(\Lambda, \kappa)$ we exactly compute the five terms on the right-hand side of (3.13) separately.

(1) We have

$$\begin{aligned}
2 \left(\Psi_3^\mu, H_0^{-1} \Psi_1^\mu \right) &= \left(\Omega, -(A^- \cdot A^-) H_0^{-1} A_\mu H_0^{-1} A_\mu^+ \Omega \right) \\
&\quad + \frac{1}{2} \left(\Omega, (A^- \cdot A^-) H_0^{-1} (P_f \cdot A + A \cdot P_f) H_0^{-1} P_{f\mu} H_0^{-1} A_\mu^+ \Omega \right).
\end{aligned} \tag{3.30}$$

Since $P_{f\mu} H_0^{-1} A_\mu \Omega = H_0^{-1} A_\mu P_{f\mu} \Omega = 0$, the second term of the right-hand side of (3.30) vanishes, we have

$$\begin{aligned}
2 \left(\Psi_3^\mu, H_0^{-1} \Psi_1^\mu \right) &= - \left(\Omega, (A^- \cdot A^-) H_0^{-1} A_\mu^+ H_0^{-1} A_\mu^+ \Omega \right) \\
&= - \iint d^3 k_1 d^3 k_2 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \frac{1}{E_{12}} \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \text{tr}(Q_1 Q_2).
\end{aligned} \tag{3.31}$$

(2) We have

$$\begin{aligned}
\left(\Psi_2^\mu, H_0^{-1} \Psi_2^\mu \right) &= \left(\frac{1}{2} \right)^2 \left(P_{f\mu} H_0^{-1} (A^+ \cdot A^+) \Omega, H_0^{-1} P_{f\mu} H_0^{-1} (A^+ \cdot A^+) \Omega \right) \\
&= \left(\frac{1}{2} \right)^2 \left(\Omega, (A^- \cdot A^-) \left(H_0^{-1} \right)^3 (P_f \cdot P_f) (A^+ \cdot A^+) \Omega \right) \\
&= \left(\frac{1}{2} \right)^2 \iint d^3 k_1 d^3 k_2 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \left(\frac{1}{E_{12}} \right)^3 |k_1 + k_2|^2 2 \text{tr}(Q_1 Q_2).
\end{aligned} \tag{3.32}$$

(3) We have

$$\begin{aligned}
&-2 \left(\Psi_2^\mu, H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu \right) \\
&= \frac{1}{2} \left(P_{f\mu} H_0^{-1} (A^+ \cdot A^+) \Omega, H_0^{-1} (P_f \cdot A + A \cdot P_f) H_0^{-1} A_\mu^+ \Omega \right) \\
&= \sum_{\nu=1,2,3} \left(\Omega, (A^- \cdot A^-) H_0^{-1} P_{f\mu} H_0^{-1} P_{f\nu} A_\nu^+ H_0^{-1} A_\mu^+ \Omega \right) \\
&= \iint d^3 k_1 d^3 k_2 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \left(\frac{1}{E_{12}} \right)^2 \left(\frac{1}{E_1} + \frac{1}{E_2} \right) (k_2, Q_1 Q_2 k_1).
\end{aligned} \tag{3.33}$$

(4) We have

$$\begin{aligned}
&-\frac{1}{2} \left(\Psi_1^\mu, H_0^{-1} H_2 H_0^{-1} \Psi_1^\mu \right) \\
&= -\frac{1}{2} \left(A_\mu^+ \Omega, H_0^{-1} ((A^+ \cdot A^+) + 2(A^+ \cdot A^-) + (A^- \cdot A^-)) H_0^{-1} A_\mu^+ \Omega \right) \\
&= - \left(\Omega, A_\mu^- H_0^{-1} (A^+ \cdot A^-) H_0^{-1} A_\mu^+ \Omega \right) \\
&= - \iint d^3 k_1 d^3 k_2 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \frac{1}{E_1} \frac{1}{E_2} \text{tr}(Q_1 Q_2).
\end{aligned} \tag{3.34}$$

(5) We have

$$\begin{aligned}
& \left(\Psi_1^\mu, H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu \right) \\
&= \left(\frac{1}{2} \right)^2 \left(A_\mu^+ \Omega, H_0^{-1} (P_f \cdot A + A \cdot P_f) H_0^{-1} (P_f \cdot A + A \cdot P_f) H_0^{-1} A_\mu^+ \Omega \right) \\
&= \left(A_\mu^+ \Omega, H_0^{-1} (P_f \cdot A) H_0^{-1} (P_f \cdot A) H_0^{-1} A_\mu^+ \Omega \right) \\
&= \sum_{\nu, \kappa=1,2,3} \left(A_\mu^+ \Omega, H_0^{-1} P_{f\nu} A_\nu^+ H_0^{-1} P_{f\kappa} A_\kappa^- H_0^{-1} A_\mu^+ \Omega \right) \\
&\quad + \sum_{\nu, \kappa=1,2,3} \left(A_\mu^+ \Omega, H_0^{-1} P_{f\nu} A_\nu^- H_0^{-1} P_{f\kappa} A_\kappa^+ H_0^{-1} A_\mu^+ \Omega \right). \quad (3.35)
\end{aligned}$$

Since

$$P_{f\kappa} A_\kappa^- H_0^{-1} A_\mu^+ \Omega = 0,$$

the first term on the last line in (3.35) vanishes. Then we have

$$\begin{aligned}
& \left(\Psi_1^\mu, H_0^{-1} H_1 H_0^{-1} H_1 H_0^{-1} \Psi_1^\mu \right) \\
&= \sum_{\nu, \kappa=1,2,3} \left(\Omega, A_\mu^- H_0^{-1} P_{f\nu} A_\nu^- H_0^{-1} P_{f\kappa} A_\kappa^+ H_0^{-1} A_\mu^+ \Omega \right) \\
&= \iint d^3 k_1 d^3 k_2 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \frac{1}{E_{12}} \left\{ \left(\frac{1}{E_2} \right)^2 2(k_2, Q_1 k_2) + \frac{1}{E_1} \frac{1}{E_2} (k_2, Q_1 Q_2 k_1) \right\} \\
&= \iint d^3 k_1 d^3 k_2 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \frac{1}{E_{12}} \left\{ \left(\frac{1}{E_1} \right)^2 (k_1, Q_2 k_1) + \left(\frac{1}{E_2} \right)^2 (k_2, Q_1 k_2) \right\} \\
&\quad + \iint d^3 k_1 d^3 k_2 \frac{|\hat{\varphi}_1|^2}{2\omega_1} \frac{|\hat{\varphi}_2|^2}{2\omega_2} \frac{1}{E_{12}} \frac{1}{E_1} \frac{1}{E_2} (k_2, Q_1 Q_2 k_1). \quad (3.36)
\end{aligned}$$

(3.28) follows from Lemma 3.6, (3.31), (3.32), (3.33), (3.34), (3.36) and the facts

$$\begin{aligned}
\text{tr}[Q(k_1)Q(k_2)] &= \sum_{j,j'=1,2} (e_j(k_1)e_{j'}(k_2))^2 = 1 + (\hat{k}_1, \hat{k}_2)^2, \\
(k_1, Q(k_2)Q(k_1)k_2) &= (k_1, k_2)((\hat{k}_1, \hat{k}_2)^2 - 1), \\
(k_1, Q(k_2)k_1) &= |k_1|^2(1 - (\hat{k}_1, \hat{k}_2)^2).
\end{aligned}$$

Thus the proof is complete. \square

4 Main theorem

By (3.29) we can see that

$$a_2(\Lambda, \kappa) = \frac{(4\pi)^2}{(2\pi)^6} \frac{2}{3} \sum_{j=1}^6 b_j(\Lambda), \quad (4.1)$$

where

$$\begin{aligned}
b_1(\Lambda) &= - \int_{-1}^1 dX (1 + X^2) \int_{\kappa}^{\Lambda} dr_1 \int_{\kappa}^{\Lambda} dr_2 \pi r_1 r_2 \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \frac{1}{F_{12}}, \\
b_2(\Lambda) &= \int_{-1}^1 dX (1 + X^2) \int_{\kappa}^{\Lambda} dr_1 \int_{\kappa}^{\Lambda} dr_2 \pi r_1 r_2 \left(\frac{1}{F_{12}} \right)^3 \frac{r_1^2 + 2r_1 r_2 X + r_2^2}{2}, \\
b_3(\Lambda) &= \int_{-1}^1 dX X (-1 + X^2) \int_{\kappa}^{\Lambda} dr_1 \int_{\kappa}^{\Lambda} dr_2 \pi r_1^2 r_2^2 \left(\frac{1}{F_1} + \frac{1}{F_2} \right) \left(\frac{1}{F_{12}} \right)^2, \\
b_4(\Lambda) &= - \int_{-1}^1 dX (1 + X^2) \int_{\kappa}^{\Lambda} dr_1 \int_{\kappa}^{\Lambda} dr_2 \pi r_1 r_2 \frac{1}{F_1} \frac{1}{F_2}, \\
b_5(\Lambda) &= \int_{-1}^1 dX (1 - X^2) \int_{\kappa}^{\Lambda} dr_1 \int_{\kappa}^{\Lambda} dr_2 \pi r_1 r_2 \left(\frac{r_1^2}{F_1^2} + \frac{r_2^2}{F_2^2} \right) \frac{1}{F_{12}}, \\
b_6(\Lambda) &= \int_{-1}^1 dX X (-1 + X^2) \int_{\kappa}^{\Lambda} dr_1 \int_{\kappa}^{\Lambda} dr_2 \pi r_1^2 r_2^2 \frac{1}{F_1} \frac{1}{F_2} \frac{1}{F_{12}}.
\end{aligned}$$

Our main theorem is stated as follows.

Theorem 4.1 *There exist strictly positive constants c_1 and c_2 such that*

$$c_1 \leq \lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda, \kappa)}{\sqrt{\Lambda}} \leq c_2.$$

To prove Theorem 4.1 we estimate the lower and upper bounds of $a_2(\Lambda, \kappa)/\sqrt{\Lambda}$ as $\Lambda \rightarrow \infty$ in what follows.

Let $\rho_{\Lambda}(\cdot, \cdot) : [0, \infty) \times [-1, 1] \rightarrow \mathbb{R}$ be defined by

$$\rho_{\Lambda} = \rho_{\Lambda}(r, X) = r^2 + 2\Lambda r X + \Lambda^2 + 2r + 2\Lambda = (r + \Lambda X + 1)^2 + \Delta,$$

where

$$\Delta = \Lambda^2(1 - X^2) + 2\Lambda(1 - X) - 1. \quad (4.2)$$

Lemma 4.2 *There exist constants C_1, C_2, C_3 and C_4 such that for sufficiently large $\Lambda > 0$,*

$$\begin{aligned}
(1) \quad & \int_{-1}^1 dX \int_0^{\Lambda} dr \frac{1}{\rho_{\Lambda}(r, X)} \leq C_1 \frac{1}{\Lambda}, \\
(2) \quad & \int_{-1}^1 dX \int_0^{\Lambda} dr \left(\frac{1}{\rho_{\Lambda}(r, X)} \right)^2 \leq C_2 \frac{1}{\Lambda^{5/2}}, \\
(3) \quad & \int_{-1}^1 dX \int_0^{\Lambda} dr \frac{1}{\rho_{\Lambda}(r, X)} \frac{1}{r+2} \leq C_3 \frac{\log \Lambda}{\Lambda^2}, \\
(4) \quad & \int_{-1}^1 dX \int_0^{\Lambda} dr \left(\frac{1}{\rho_{\Lambda}(r, X)} \right)^2 (1 - X^2) \leq C_4 \frac{1}{\Lambda^3}.
\end{aligned}$$

Proof: See Appendix A. □

4.1 Upper bounds

Lemma 4.3 *There exists a constant C_{\max} such that*

$$\lim_{\Lambda \rightarrow \infty} \left| \frac{a(\Lambda, \kappa)}{\sqrt{\Lambda}} \right| < C_{\max}.$$

Proof: Note that for a continuous function f ,

$$\frac{d}{d\Lambda} \int_{\kappa}^{\Lambda} dr_1 \int_{\kappa}^{\Lambda} dr_2 f(r_1, r_2) = \int_{\kappa}^{\Lambda} f(\Lambda, r) dr + \int_{\kappa}^{\Lambda} f(r, \Lambda) dr. \quad (4.3)$$

In this proof, C denotes some sufficiently large constant and is not necessarily the same number.

(1) We have

$$\frac{d}{d\Lambda} b_1(\Lambda) = 8\pi \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr \frac{1}{\rho_{\Lambda}(r, X)} \left(\frac{\Lambda}{r+2} + \frac{r}{\Lambda+2} \right) (1 + X^2).$$

Since by Lemma 4.2 (3) and (1),

$$\begin{aligned} \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr \frac{1}{\rho_{\Lambda}(r, X)} \frac{\Lambda}{r+2} &\leq C \frac{\log \Lambda}{\Lambda}, \\ \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr \frac{1}{\rho_{\Lambda}(r, X)} \frac{r}{\Lambda+2} &\leq C \frac{1}{\Lambda}, \end{aligned}$$

we have

$$\left| \frac{d}{d\Lambda} b_1(\Lambda) \right| \leq C \frac{\log \Lambda}{\Lambda}. \quad (4.4)$$

(2) We see that by Lemma 4.2 (2),

$$\begin{aligned} \frac{d}{d\Lambda} b_2(\Lambda) &= 8\pi \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr \left(\frac{1}{\rho_{\Lambda}(r, X)} \right)^3 r \Lambda (\Lambda^2 + r^2 + 2\Lambda r X) (1 + X^2) \\ &\leq 8\pi \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr \left(\frac{1}{\rho_{\Lambda}(r, X)} \right)^2 r \Lambda (1 + X^2) \\ &\leq 8\pi \Lambda^2 \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr \left(\frac{1}{\rho_{\Lambda}(r, X)} \right)^2 (1 + X^2) \\ &\leq C \frac{1}{\sqrt{\Lambda}}. \end{aligned}$$

Hence

$$\left| \frac{d}{d\Lambda} b_2(\Lambda) \right| \leq C \frac{1}{\sqrt{\Lambda}}. \quad (4.5)$$

(3) We have

$$\left| \frac{d}{d\Lambda} b_3(\Lambda) \right| = 16\pi \left| \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr X (X^2 - 1) \left(\frac{1}{\rho_{\Lambda}(r, X)} \right)^2 \left(\frac{\Lambda^2 r}{r+2} + \frac{r^2 \Lambda}{\Lambda+2} \right) \right|.$$

Since by Lemma 4.2 (3) and (4),

$$\begin{aligned} \left| \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr X (X^2 - 1) \left(\frac{1}{\rho_{\Lambda}(r, X)} \right)^2 \frac{\Lambda^2 r}{r+2} \right| &\leq C \frac{1}{\Lambda}, \\ \left| \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr X (X^2 - 1) \left(\frac{1}{\rho_{\Lambda}(r, X)} \right)^2 \frac{r^2 \Lambda}{\Lambda+2} \right| &\leq C \frac{1}{\Lambda}, \end{aligned}$$

we have

$$\left| \frac{d}{d\Lambda} b_3(\Lambda) \right| \leq C \frac{1}{\Lambda}. \quad (4.6)$$

(4) It is trivial that

$$|b_4(\Lambda)| \leq C[\log \Lambda]^2. \quad (4.7)$$

(5) We have

$$\frac{d}{d\Lambda} b_5(\Lambda) = 16\pi \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr \frac{1}{\rho_{\Lambda}(r, X)} (1 - X^2) r \Lambda \left\{ \left(\frac{1}{\Lambda+2} \right)^2 + \left(\frac{1}{r+2} \right)^2 \right\}.$$

Since by Lemma 4.2 (1),

$$\begin{aligned} \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr \frac{1}{\rho_{\Lambda}(r, X)} (1 - X^2) r \Lambda \left(\frac{1}{\Lambda+2} \right)^2 &\leq C \frac{1}{\Lambda}, \\ \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr \frac{1}{\rho_{\Lambda}(r, X)} (1 - X^2) r \Lambda \left(\frac{1}{r+2} \right)^2 &\leq C \frac{1}{\Lambda}, \end{aligned}$$

we see that

$$\left| \frac{d}{d\Lambda} b_5(\Lambda) \right| \leq C \frac{1}{\Lambda}. \quad (4.8)$$

(6) We have by Lemma 4.2 (1)

$$\left| \frac{d}{d\Lambda} b_6(\Lambda) \right| = 16\pi \left| \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr (X^2 - 1) X \frac{1}{\rho_{\Lambda}(r, X)} \frac{r}{r+2} \frac{\Lambda}{\Lambda+2} \right| \leq C \frac{1}{\Lambda}.$$

Then we have

$$\left| \frac{d}{d\Lambda} b_6(\Lambda) \right| \leq C \frac{1}{\Lambda}. \quad (4.9)$$

From (4.4)-(4.9) it follows that

$$\begin{aligned} |b_j(\Lambda)| &\leq C[\log \Lambda]^2, \quad j = 1, 4, \\ |b_2(\Lambda)| &\leq C\Lambda^{1/2}, \quad |b_j(\Lambda)| \leq C \log \Lambda, \quad j = 3, 5, 6. \end{aligned} \quad (4.10)$$

Then the lemma follows. \square

4.2 Lower bounds

Lemma 4.4 *There exists a positive constant $C_{\min} > 0$ such that*

$$C_{\min} \leq \lim_{\Lambda \rightarrow \infty} \frac{b_2(\Lambda)}{\sqrt{\Lambda}}.$$

From this lemma and (4.10), i.e.,

$$\lim_{\Lambda \rightarrow \infty} \frac{b_j(\Lambda)}{\sqrt{\Lambda}} = 0, \quad j = 1, 3, 4, 5, 6,$$

the following corollary follows.

Corollary 4.5 *It follows that*

$$C_{\min} \leq \lim_{\Lambda \rightarrow \infty} \frac{a_2(\Lambda, \kappa)}{\sqrt{\Lambda}}.$$

Proof of Lemma 4.4: We have

$$\frac{d}{d\Lambda} b_2(\Lambda) = 8\pi \int_{-1}^1 dX \int_{\kappa}^{\Lambda} dr (1 + X^2) \left(\frac{1}{\rho_{\Lambda}(r, X)} \right)^3 (r^2 + 2r\Lambda X + \Lambda^2) r \Lambda, \quad (4.11)$$

where, recall that

$$\begin{aligned} \rho_{\Lambda}(r, X) &= (r + \Lambda X + 1)^2 + \Delta, \\ \Delta &= \Lambda^2(1 - X^2) + \Lambda(1 - X) - 1. \end{aligned}$$

Note that $\Delta > 0$ for $X \leq 0$ and a sufficiently large Λ . Since the integrand of (4.11)

$$T_R(r) = \left(\frac{1}{\rho_{\Lambda}(r, X)} \right)^3 (r^2 + 2r\Lambda X + \Lambda^2) r \Lambda$$

is positive, it is enough to prove that

$$\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \frac{d}{d\Lambda} b_2(\Lambda) = \lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \int_{-1+(1/\Lambda)}^0 dX \int_{\kappa}^{\Lambda} dr T_R(r) (1 + X^2) > 0. \quad (4.12)$$

We simply set $\rho = \rho_{\Lambda}(r, X)$. Since

$$(r^2 + 2r\Lambda X + \Lambda^2) r \Lambda = \Lambda \left\{ (r - 2)\rho + (4 + 4\Lambda X - 2\Lambda)r + 2(\Lambda^2 + 2\Lambda) \right\},$$

we have

$$\begin{aligned} \int_{\kappa}^{\Lambda} dr T_R(r) &= \Lambda \int_{\kappa}^{\Lambda} dr \frac{r - 2}{\rho^2} + \Lambda(4 + 4\Lambda X - 2\Lambda) \int_{\kappa}^{\Lambda} dr \frac{r}{\rho^3} + 2\Lambda(\Lambda^2 + 2\Lambda) \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^3} \\ &= \Lambda \int_{\kappa}^{\Lambda} dr \frac{r}{\rho^2} + \Lambda^2(4X - 2) \int_{\kappa}^{\Lambda} dr \frac{r}{\rho^3} + 2\Lambda^3 \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^3} + t_1(\Lambda), \end{aligned}$$

where

$$t_1(\Lambda) = -2\Lambda \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^2} + 4\Lambda \int_{\kappa}^{\Lambda} dr \frac{r}{\rho^3} + 2\Lambda^2 \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^3}.$$

Moreover since

$$\begin{aligned} \int_{\kappa}^{\Lambda} dr \frac{r}{\rho^2} &= -\frac{1}{2} \left[\frac{1}{\rho} \right]_{\kappa}^{\Lambda} - \int_{\kappa}^{\Lambda} dr \frac{\Lambda X + 1}{\rho^2}, \\ \int_{\kappa}^{\Lambda} dr \frac{r}{\rho^3} &= -\frac{1}{4} \left[\frac{1}{\rho^2} \right]_{\kappa}^{\Lambda} - \int_{\kappa}^{\Lambda} dr \frac{\Lambda X + 1}{\rho^3}, \end{aligned}$$

we have

$$\int_{\kappa}^{\Lambda} dr T_R(r) = -\Lambda^2 X \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^2} + \Lambda^3 (2 - X(4X - 2)) \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^3} + t_1(\Lambda) + t_2(\Lambda),$$

where

$$\begin{aligned} t_2(\Lambda) &= \Lambda \left(-\frac{1}{2}\right) \left[\frac{1}{\rho} \right]_{\kappa}^{\Lambda} - \Lambda \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^2} \\ &\quad + \Lambda^2 (4X - 2) \left(-\frac{1}{4}\right) \left[\frac{1}{\rho^2} \right]_{\kappa}^{\Lambda} - \Lambda^2 (4X - 2) \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^3}. \end{aligned}$$

Note that

$$\int \frac{1}{(x^2 + a^2)^{n+1}} dx = \frac{1}{2na^2} \frac{x}{(x^2 + a^2)^n} + \frac{2n-1}{2n} \frac{1}{a^2} \int \frac{1}{(x^2 + a^2)^n} dx, \quad n \geq 1.$$

Then

$$\begin{aligned} \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^2} &= \left[\frac{r + \Lambda X + 1}{2\Delta} \frac{1}{\rho} \right]_{\kappa}^{\Lambda} + \frac{1}{2\Delta^{3/2}} \left[\arctan \frac{r + \Lambda X + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda}, \\ \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^3} &= \left[\frac{r + \Lambda X + 1}{4\Delta} \frac{1}{\rho^2} \right]_{\kappa}^{\Lambda} + \frac{3}{8} \left[\frac{r + \Lambda X + 1}{\Delta^2} \frac{1}{\rho} \right]_{\kappa}^{\Lambda} \\ &\quad + \frac{3}{8} \frac{1}{\Delta^{5/2}} \left[\arctan \frac{r + \Lambda X + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda}. \end{aligned}$$

Hence we have

$$\begin{aligned} \int_{\kappa}^{\Lambda} dr T_R(r) &= -\Lambda^2 X \frac{1}{2\Delta^{3/2}} \left\{ \left[\arctan \frac{r + \Lambda X + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda} + \left[\sqrt{\Delta} \frac{r + \Lambda X + 1}{\rho} \right]_{\kappa}^{\Lambda} \right\} \\ &\quad + \frac{3}{8} \Lambda^3 2(2X + 1)(1 - X) \frac{1}{\Delta^{5/2}} \left\{ \left[\arctan \frac{r + \Lambda X + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda} + \left[\sqrt{\Delta} \frac{r + \Lambda X + 1}{\rho} \right]_{\kappa}^{\Lambda} \right\} \\ &\quad + t_1(\Lambda) + t_2(\Lambda) + t_3(\Lambda), \end{aligned}$$

where

$$t_3(\Lambda) = \Lambda^3 2(2X+1)(1-X) \left[\frac{r + \Lambda X + 1}{4\Delta} \frac{1}{\rho^2} \right]_{\kappa}^{\Lambda}.$$

It is proven in Lemma B.1 of Appendix B that

$$\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \int_{-1+1/\Lambda}^0 (1+X^2)(t_1(\Lambda) + t_2(\Lambda) + t_3(\Lambda)) dX = 0. \quad (4.13)$$

From this it is enough to show that

$$\begin{aligned} \lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \Lambda^2 \int_{-1+1/\Lambda}^0 dX (1+X^2) \frac{1}{\Delta^{3/2}} \left[\sqrt{\Delta} \frac{r + \Lambda X + 1}{\rho} \right]_{\kappa}^{\Lambda} \times \\ \times \frac{1}{4} \left\{ -2X + 3(1-X)(2X+1) \frac{\Lambda}{\Delta} \right\} \geq 0 \end{aligned} \quad (4.14)$$

and that there exists a positive constant $\xi > 0$ such that

$$\begin{aligned} \lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \int_{-1+1/\Lambda}^0 dX (1+X^2) \left\{ -\Lambda^2 X \frac{1}{2\Delta^{3/2}} \left[\arctan \frac{r + \Lambda X + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda} \right. \\ \left. + \frac{3}{8} \Lambda^3 2(2X+1)(1-X) \frac{1}{\Delta^{5/2}} \left[\arctan \frac{r + \Lambda X + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda} \right\} \\ = \lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \Lambda^2 \int_{-1+1/\Lambda}^0 dX (1+X^2) \frac{1}{\Delta^{3/2}} \left[\arctan \frac{r + \Lambda X + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda} \times \\ \times \frac{1}{4} \left\{ -2X + 3(1-X)(2X+1) \frac{\Lambda}{\Delta} \right\} > \xi \end{aligned} \quad (4.15)$$

Changing variable X to $-y$, we shall prove (4.15), i.e.,

$$\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \Lambda^2 \int_0^{1-(1/\Lambda)} dy (1+y^2) \frac{1}{\Delta^{3/2}} \left[\arctan \frac{r - \Lambda y + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda} \frac{1}{4} a_{\Lambda}(y) dy > \xi,$$

where

$$\begin{aligned} a_{\Lambda}(y) &= 2y + \frac{6}{\Lambda} + b_{\Lambda}(y), \\ b_{\Lambda}(y) &= \frac{3}{\Lambda} \left(1 + \frac{2}{\Lambda} \right) \frac{y + \frac{2\Lambda+3}{2\Lambda+4}}{\left(y - \frac{1}{2\Lambda} \right)^2 - \frac{(2\Lambda+3)(2\Lambda-1)}{4\Lambda^2}}. \end{aligned}$$

The function $b_{\Lambda}(\cdot)$ satisfies the following properties:

(1) $b_{\Lambda}''(y) < 0$ for $0 \leq y \leq 1 - 1/\Lambda$, i.e., $b_{\Lambda}(y)$ is concave for $0 \leq y \leq 1 - 1/\Lambda$,

(2) $\lim_{\Lambda \rightarrow \infty} b_{\Lambda}(1 - 1/\Lambda) = -3/2$,

(3) $\lim_{\Lambda \rightarrow \infty} b_{\Lambda}(y) = 0$ for $y \neq 1$ and $\lim_{\Lambda \rightarrow \infty} b_{\Lambda}(1) = -3$.

By (1)–(3) we have

$$\inf_{0 \leq y \leq 1-1/\Lambda} b_{\Lambda}(y) = \min \{b_{\Lambda}(0), b_{\Lambda}(1 - 1/\Lambda)\}$$

and then for sufficiently large Λ ,

$$\inf_{0 \leq y \leq 1-1/\Lambda} b_{\Lambda}(y) = b_{\Lambda}(1 - 1/\Lambda) > -\frac{7}{4}.$$

Hence

$$\inf_{15/16 \leq y \leq 1-1/\Lambda} a_{\Lambda}(y) \geq \frac{15}{8} - \frac{7}{4} = \frac{1}{8} > 0.$$

Moreover

$$\begin{aligned} \left[\arctan \frac{r - \Lambda y + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda} &= \arctan \frac{(1-y)\Lambda + 1}{\sqrt{\Delta}} - \arctan \frac{\kappa - \Lambda y + 1}{\sqrt{\Delta}} > 0, \\ \lim_{\Lambda \rightarrow \infty} \left[\arctan \frac{r - \Lambda y + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda} &= \arctan \frac{1-y}{\sqrt{1-y^2}} + \arctan \frac{y}{\sqrt{1-y^2}} > 0 \end{aligned}$$

for $0 \leq y \leq 1$. Then

$$\delta = \inf_{\Lambda > 1} \inf_{0 \leq y \leq 1} \left[\arctan \frac{r - \Lambda y + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda} > 0$$

Then we have

$$\begin{aligned} &\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \Lambda^2 \int_0^{1-(1/\Lambda)} dy (1+y^2) \frac{1}{\Delta^{3/2}} \left[\arctan \frac{r - \Lambda y + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda} \frac{1}{4} a_{\Lambda}(y) \\ &\geq \lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \Lambda^2 \int_{15/16}^{1-(1/\Lambda)} dy (1+y^2) \frac{1}{\Delta^{3/2}} \left[\arctan \frac{r - \Lambda y + 1}{\sqrt{\Delta}} \right]_{\kappa}^{\Lambda} \frac{1}{4} a_{\Lambda}(y) \\ &\geq \frac{1}{8} \times \frac{1}{4} \times \delta \times \lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \Lambda^2 \int_{15/16}^{1-(1/\Lambda)} dy \frac{1}{\Delta^{3/2}}. \end{aligned}$$

Furthermore

$$\begin{aligned} &\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \Lambda^2 \int_{15/16}^{1-(1/\Lambda)} dy \frac{1}{\Delta^{3/2}} \\ &= \lim_{\Lambda \rightarrow \infty} \frac{1}{\sqrt{\Lambda}} \int_{15/16}^{1-(1/\Lambda)} dy \frac{1}{\left\{ (1-y^2) + \frac{1}{\Lambda}(1+y) - \frac{1}{\Lambda^2} \right\}^{3/2}} \end{aligned}$$

$$\begin{aligned}
&\geq \lim_{\Lambda \rightarrow \infty} \frac{1}{\sqrt{\Lambda}} \int_{15/16}^{1-(1/\Lambda)} dy \frac{1}{\left\{(1-y) + \frac{1}{\Lambda}\right\}^{3/2}} \frac{1}{(1+y)^{3/2}} \\
&\geq \lim_{\Lambda \rightarrow \infty} \frac{1}{\sqrt{\Lambda}} \int_{15/16}^{1-(1/\Lambda)} dy \frac{1}{\left\{(1-y) + \frac{1}{\Lambda}\right\}^{3/2}} \frac{1}{2^{3/2}} \\
&= \lim_{\Lambda \rightarrow \infty} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\Lambda}} \left(\frac{1}{\sqrt{2/\Lambda}} - \frac{1}{\sqrt{1/16 + 1/\Lambda}} \right) = \frac{1}{2}.
\end{aligned}$$

Then we proved that

$$\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \frac{d}{d\Lambda} b_2(\Lambda) > 4\pi \times \frac{1}{8} \times \frac{1}{4} \times \frac{1}{2} \times \delta = \frac{\pi\delta}{16} > 0.$$

Then (4.15) follows. We shall show (4.14). Since the left-hand side of (4.14) is

$$\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \Lambda^2 \int_0^{1-(1/\Lambda)} dy (1+y^2) \frac{1}{\Delta^{3/2}} \left[\sqrt{\Delta} \frac{r - \Lambda y + 1}{\rho} \right]_{\kappa}^{\Lambda} \frac{1}{4} a_{\Lambda}(y) dy, \quad (4.16)$$

it is enough to show that $[\cdot \cdot \cdot]_{\kappa}^{\Lambda}$ in (4.16) is nonnegative. We can directly see that

$$\left[\sqrt{\Delta} \frac{r - \Lambda y + 1}{\rho} \right]_{\kappa}^{\Lambda} = \frac{\sqrt{\Delta} K}{\{(\Lambda + \Lambda X + 1)^2 + \Delta\} \{(\kappa + \Lambda X + 1)^2 + \Delta\}},$$

where, for $0 \leq y \leq 1$,

$$K = (-2y^2 + y + 1)\Lambda^3 + (1 + 4y)\Lambda^2 - 2\Lambda$$

$$+ \kappa((y^2 - 2)\Lambda^2 + (-2y - 2)\Lambda + 1) + \kappa^2((-y + 1)\Lambda + 1).$$

Since $K > 0$ for a sufficiently large Λ , (4.14) follows. \square

Proof of Theorem 4.1: The theorem follows from Lemma 4.3 and Corollary 4.5. \square

Remark 4.6 (1) $a_2(\Lambda, \kappa)/\sqrt{\Lambda}$ converges to a nonnegative constant as $\Lambda \rightarrow \infty$.

(2) By (4.1), we can define $a_2(\Lambda, 0)$ since $b_j(\Lambda)$ with $\kappa = 0$ are finite. Moreover $a_2(\Lambda, 0)$ also satisfies Theorem 4.1.

Appendix

A Proof of Lemma 4.2

Proof of Lemma 4.2

By the definition of Δ it follows that for sufficiently large Λ ,

$$\frac{1}{\Delta} \leq \frac{1}{\Lambda}, \quad \text{for } X \leq 0.$$

Let

$$\delta = \delta(k) = \frac{1}{\Lambda^k}, \quad 0 < k \leq 2.$$

Then for sufficiently large Λ ,

$$\Delta \geq \Lambda^2(1 - X^2) - 1 > 0, \quad \text{for } -1 + \delta(k) < X \leq 0, \quad 0 < k \leq 2.$$

In particular we obtain

$$\frac{1}{\Delta} \leq \frac{a}{\Lambda^2} \frac{1}{1 - X^2}, \quad \text{for } -1 + \delta(k) \leq X \leq 0, \quad 0 < k \leq 2,$$

with some constant a independent of Λ . In this proof C denotes some sufficiently large constant and it is not necessarily the same number. We divide $\int_{-1}^1 \cdots dX$ as

$$\int_{-1}^1 \cdots dX = \int_0^1 + \int_{-1+\delta}^0 + \int_{-1}^{-1+\delta}.$$

(1) It is trivial that $\left| \int_0^1 \cdots dX \right| \leq \frac{C}{\Lambda}$. Note that

$$\begin{aligned} \left| \int_0^\Lambda dr \frac{1}{\rho_\Lambda(r, X)} \right| &= \frac{1}{\sqrt{\Delta}} \left| \arctan \frac{\Lambda + \Lambda X + 1}{\sqrt{\Delta}} - \arctan \frac{\Lambda X + 1}{\sqrt{\Delta}} \right| \\ &\leq \pi \frac{1}{\sqrt{\Delta}}. \end{aligned}$$

Let $\delta = \delta(1/2) = 1/\sqrt{\Lambda}$. Hence we have

$$\begin{aligned} \left| \int_{-1+\delta}^0 \cdots dX \right| &\leq \frac{C}{\Lambda} \arcsin(1 - \delta), \\ \left| \int_{-1}^{-1+\delta} \cdots dX \right| &\leq \frac{C}{\sqrt{\Lambda}} \delta. \end{aligned}$$

Thus (1) follows.

(2) It is trivial that $\left| \int_0^1 \cdots dX \right| \leq \frac{C}{\Lambda^3}$. Note that

$$\begin{aligned} & \left| \int_0^\Lambda dr \frac{1}{\rho_\Lambda(r, X)^2} \right| \\ &= \frac{1}{2\Delta} \left| \int_0^\Lambda \frac{1}{\rho_\Lambda(r, X)} dr + \left(\frac{\Lambda + \Lambda X + 1}{(\Lambda + \Lambda X + 1)^2 + \Delta} - \frac{\Lambda X + 1}{(\Lambda X + 1)^2 + \Delta} \right) \right| \\ &\leq \begin{cases} \frac{C}{\Lambda^2} \left(\frac{1}{\Lambda(1-X^2)^{3/2}} + \frac{1}{\Lambda} \right), & -1 + \delta \leq X \leq 0, \\ \frac{C}{\Lambda} \left(\frac{1}{\sqrt{\Lambda}} + \frac{1}{\Lambda} \right), & -1 \leq X \leq -1 + \delta. \end{cases} \end{aligned}$$

Let $\delta = \delta(1) = 1/\Lambda$. Hence we have

$$\begin{aligned} \left| \int_{-1+\delta}^0 \cdots dX \right| &\leq \frac{C}{\Lambda^3} \int_{-1+\delta}^0 dX \left(\frac{1}{(1-X^2)^{3/2}} + 1 \right) \leq \frac{C}{\Lambda^3} \left(\frac{1}{\sqrt{\delta}} + 1 \right) \\ \left| \int_{-1}^{-1+\delta} \cdots dX \right| &\leq \frac{C}{\Lambda} \left(\frac{1}{\sqrt{\Lambda}} + \frac{1}{\Lambda} \right) \delta. \end{aligned}$$

Then (2) follows.

(3) We see that

$$\frac{1}{\rho_\Lambda(r, X)} \frac{1}{r+2} = \frac{l_1}{r+2} + \frac{l_2}{\rho_\Lambda(r, X)},$$

where

$$l_1 = \frac{1}{\Lambda^2} \frac{1}{(4X-1)/\Lambda-1}, \quad l_2 = \frac{1}{\Lambda^2} \frac{r+2\Lambda X}{(4X-1)/\Lambda-1}.$$

We have

$$\begin{aligned} \left| \int_{-1}^1 dX \int_0^\Lambda dr \frac{l_1}{r+2} \right| &\leq \frac{\log(\Lambda+2)}{\Lambda^2} \int_{-1}^1 dX \frac{1}{(4X-1)/\Lambda-1} \leq C \frac{\log \Lambda}{\Lambda^2}, \\ \left| \int_{-1}^1 dX \int_0^\Lambda dr \frac{l_2}{\rho_\Lambda(r, X)} \right| &\leq \frac{\Lambda}{\Lambda^2} \int_{-1}^1 dX \int_0^\Lambda \frac{1}{\rho_\Lambda(r, X)} \frac{1+2X}{(4X-1)/\Lambda-1} \leq \frac{C}{\Lambda^2}. \end{aligned}$$

Hence (3) follows.

(4) It is trivial that $\left| \int_0^1 \cdots dX \right| \leq \frac{C}{\Lambda^3}$. Let $\delta = \delta(3/2) = 1/\Lambda^{3/2}$. From the proof of (2) it follows that

$$\begin{aligned} \left| \int_{-1+\delta}^0 \cdots dX \right| &\leq \frac{C}{\Lambda^3} \int_{-1+\delta}^0 dX \left\{ \frac{(1-X^2)}{(1-X^2)^{3/2}} + (1-X^2) \right\} \\ &\leq \frac{C}{\Lambda^3} (\arcsin(1-\delta) + 1), \\ \left| \int_{-1}^{-1+\delta} \cdots dX \right| &\leq \frac{C}{\Lambda} \left(\frac{1}{\sqrt{\Lambda}} + \frac{1}{\Lambda} \right) \delta. \end{aligned}$$

Hence (4) follows. \square

B Proof of (4.13)

Lemma B.1 *We have*

$$\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \int_{-1+1/\Lambda}^0 (1+X^2)(t_1(\Lambda) + t_2(\Lambda) + t_3(\Lambda)) dX = 0, \quad (2.1)$$

where

$$\begin{aligned} t_1(\Lambda) &= -2\Lambda \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^2} + 4\Lambda \int_{\kappa}^{\Lambda} dr \frac{r}{\rho^3} + 2\Lambda^2 \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^3}, \\ t_2(\Lambda) &= \Lambda \left(-\frac{1}{2}\right) \left[\frac{1}{\rho}\right]_{\kappa}^{\Lambda} - \Lambda \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^2}, \\ t_3(\Lambda) &= \Lambda^3 2(2X+1)(1-X) \left[\frac{r+\Lambda X+1}{4\Delta} \frac{1}{\rho^2}\right]_{\kappa}^{\Lambda}. \end{aligned}$$

Proof: In this proof C also denotes some sufficiently large constant, which is not necessarily the same number. We have

$$\begin{aligned} \sqrt{\Lambda} \Lambda \left| \int_{-1+(1/\Lambda)}^0 dX \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^2} \right| &\leq C \sqrt{\Lambda} \Lambda \frac{1}{\Lambda^{5/2}} = C \frac{1}{\Lambda}, \\ \sqrt{\Lambda} \Lambda \left| \int_{-1+(1/\Lambda)}^0 dX \int_{\kappa}^{\Lambda} dr \frac{r}{\rho^3} \right| &\leq C \sqrt{\Lambda} \Lambda^2 \frac{1}{\Lambda^{7/2}} = C \frac{1}{\Lambda}, \\ \sqrt{\Lambda} \Lambda^2 \left| \int_{-1+(1/\Lambda)}^0 dX \int_{\kappa}^{\Lambda} dr \frac{1}{\rho^3} \right| &\leq C \sqrt{\Lambda} \Lambda^2 \frac{1}{\Lambda^{7/2}} = C \frac{1}{\Lambda}. \end{aligned}$$

Then

$$\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \int_{-1+1/\Lambda}^0 dX (1+X^2) t_1(\Lambda) = 0$$

follows. Next we shall show that

$$\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \int_{-1+1/\Lambda}^0 dX (1+X^2) t_2(\Lambda) = 0. \quad (2.2)$$

Note that

$$\left| \left[\frac{1}{\rho} \right]_{\kappa}^{\Lambda} \right| = \left| \frac{1}{(\Lambda + \Lambda X + 1)^2 + \Delta} - \frac{1}{(\kappa + \Lambda X + 1)^2 + \Delta} \right| \leq \frac{2}{\Delta} \leq \frac{2}{\Lambda^2} \frac{1}{1-X^2}.$$

Then

$$\left| \int_{-1+1/\Lambda}^0 dX \left[\frac{1}{\rho} \right]_{\kappa}^{\Lambda} \right| \leq C \frac{\log \Lambda}{\Lambda^2}.$$

Similarly we can see that

$$\left| \int_{-1+1/\Lambda}^0 dX \left[\frac{1}{\rho^2} \right]_{\kappa}^{\Lambda} \right| \leq C \frac{1}{\Lambda^3},$$

which implies that

$$\begin{aligned} \sqrt{\Lambda} \Lambda \left| \int_{-1+1/\Lambda}^0 dX \left[\frac{1}{\rho} \right]_{\kappa}^{\Lambda} \right| &\leq C \frac{\log \Lambda}{\sqrt{\Lambda}}, \\ \sqrt{\Lambda} \Lambda^2 \left| \int_{-1+1/\Lambda}^0 dX \left[\frac{1}{\rho^2} \right]_{\kappa}^{\Lambda} \right| &\leq C \frac{1}{\sqrt{\Lambda}}. \end{aligned}$$

Hence (2.2) follows. Finally we shall show that

$$\lim_{\Lambda \rightarrow \infty} \sqrt{\Lambda} \int_{-1+1/\Lambda}^0 dX (1 + X^2) t_3(\Lambda) = 0. \quad (2.3)$$

We divide $\int_{-1+1/\Lambda}^0 dX$ as

$$\int_{-1+1/\Lambda}^0 dX = \int_{-1+1/\Lambda}^{-1/2} dX + \int_{-1/2}^0 dX.$$

Since

$$\frac{1}{\Delta} \leq \frac{1}{\Lambda^2} \frac{1}{1 - X^2} \leq \frac{1}{\Lambda^2} \frac{4}{3}, \quad \text{for } -\frac{1}{2} \leq X \leq 0,$$

we see that

$$\sqrt{\Lambda} \Lambda^3 \left| \int_{-1/2}^0 dX \left[\frac{r + \Lambda X + 1}{\Delta} \frac{1}{\rho^2} \right]_{\kappa}^{\Lambda} \right| \leq C \sqrt{\Lambda} \Lambda^3 \frac{\Lambda}{\Lambda^2} \frac{1}{\Lambda^3} = C \frac{1}{\sqrt{\Lambda}}. \quad (2.4)$$

On the other hand

$$\begin{aligned} &\left[\frac{r + \Lambda X + 1}{\Delta} \frac{1}{\rho^2} \right]_{\kappa}^{\Lambda} \\ &= \frac{\Lambda + \Lambda X + 1}{\Delta} \frac{1}{\{(\Lambda + \Lambda X + 1)^2 + \Delta\}^2} - \frac{\kappa + \Lambda X + 1}{\Delta} \frac{1}{\{(\kappa + \Lambda X + 1)^2 + \Delta\}^2}. \end{aligned}$$

Since

$$\left| \frac{\Lambda + \Lambda X + 1}{\Delta} \right| \leq \frac{C}{\Lambda}, \quad \left| \frac{\kappa + \Lambda X + 1}{(\kappa + \Lambda X + 1)^2 + \Delta} \right| \leq \frac{C}{\Lambda},$$

we have

$$\left| \int_{-1+1/\Lambda}^{-1/2} dX \left[\frac{r + \Lambda X + 1}{\Delta} \frac{1}{\rho^2} \right]_{\kappa}^{\Lambda} \right| \leq C \frac{1}{\Lambda} \int_{-1+1/\Lambda}^{-1/2} \frac{1}{\Delta^2} \leq C \frac{1}{\Lambda^4}.$$

Then we obtain that

$$\sqrt{\Lambda}\Lambda^3 \left| \int_{-1+1/\Lambda}^{-1/2} dX \left[\frac{r + \Lambda X + 1}{\Delta} \frac{1}{\rho^2} \right]_{\kappa}^{\Lambda} \right| \leq C \frac{1}{\sqrt{\Lambda}}. \quad (2.5)$$

Thus (2.3) follows from (2.4) and (2.5). \square

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